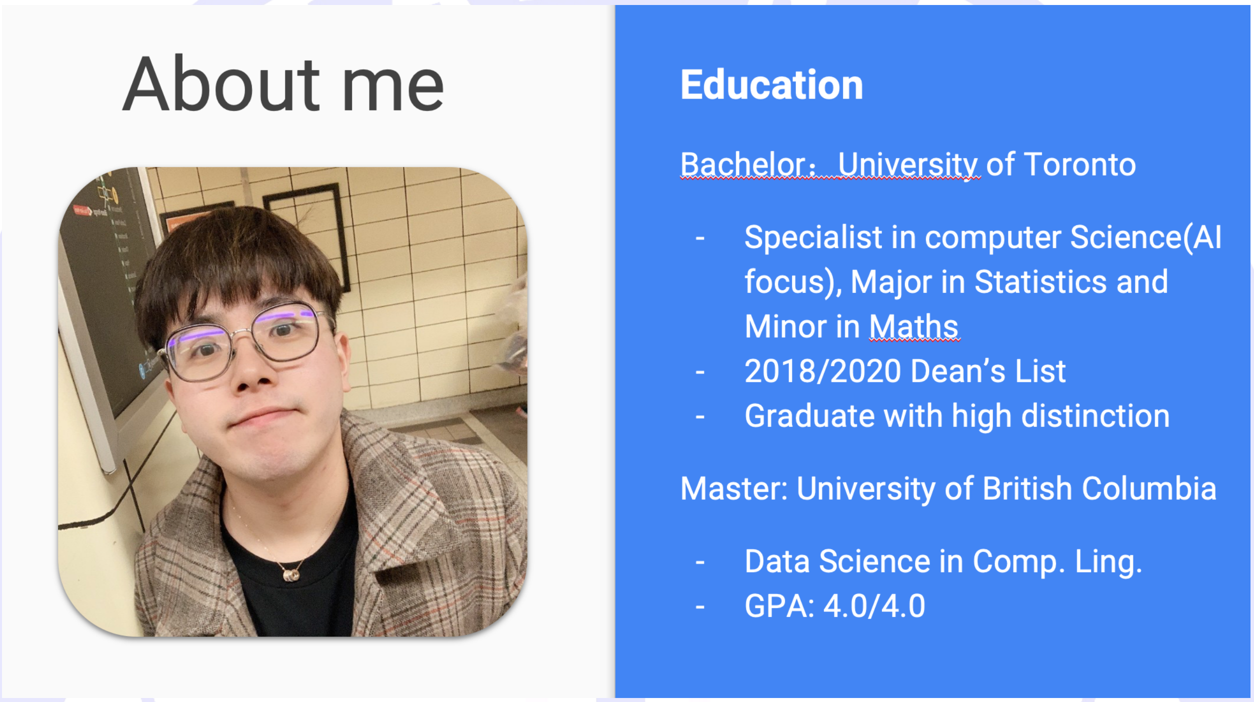
MAT137

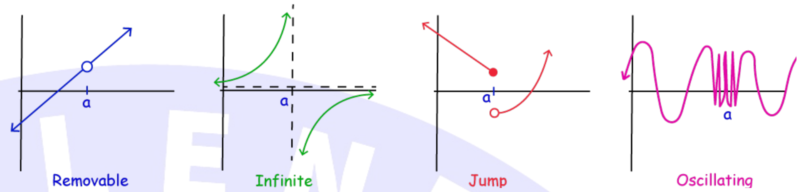
Jeremy Zhang



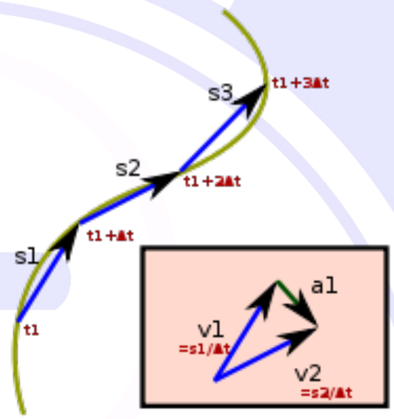
- 2018.05 - 2019.08 Software Engineer
  - Health industry tech company web and mobile app development
- 2019.09 - 2020.05 Research assistant
  - Open patent data scraping and trend modelling
- 2020.07 - present Data Science developer
  - Threat Intelligence research and big data processing

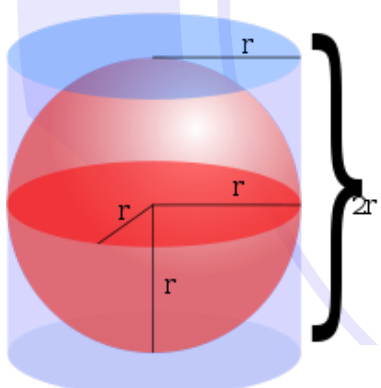
Course Overview

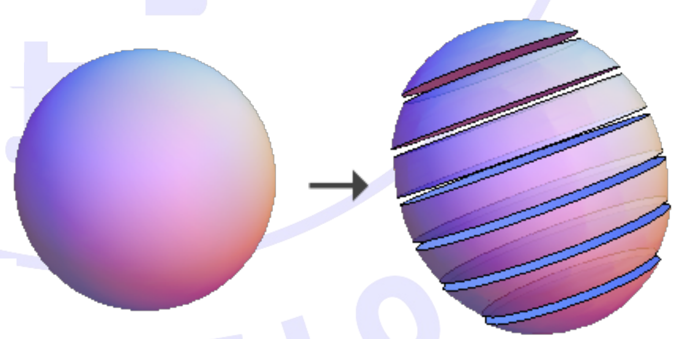
Unit 1: Logic, sets, notation, definitions, and proofs
Unit 2: Limits and continuity
Unit 3: Derivatives
Unit 4: Transcendental functions



Unit 5: The Mean Value Theorem and its applications
Unit 6: Applications of limits and derivatives
Unit 7: The definition of integral
Unit 8: The Fundamental Theorem of Calculus
Unit 9: Integration methods
Unit 10: Applications of the integral
Unit 11: Sequences
Unit 12: Improper integrals
Unit 13: Series
Unit 14: Power series and Taylor series







<u>如何选择大一数学基础课？</u>

- MAT133Y1 - ‘Calculus and linear algebra for commerce’. (一般**只有商科**学生选, **min 70% for Stats Minor**)
- MAT135H1 + MAT136H1 - ‘Calculus 1 + Calculus 2’ (the default course for most science students, **注重计算和应用，没有证明**)
- MAT137Y1 - ‘Calculus with proofs’, including an introduction to proofs and abstraction（**注重培养数学思想和证明**，会比 135 和 136 难）
- MAT157Y1 - ‘Analysis I’, intended for math specialists.（纯数学，少计算，**多证明**，和 137 比起来更抽象，**想学数学 specialist** 或者其他领域和数学结合的 specialist 的同学可以选）

<u>注意：可以选比专业要求更难的数学课, 但不能选更简单的</u>

例如 life-sci 可选 mat137，但不可选 mat133
对于 stats，computer science，推荐选 135+136/137
对于想要继续读 math 的同学，推荐 137/157

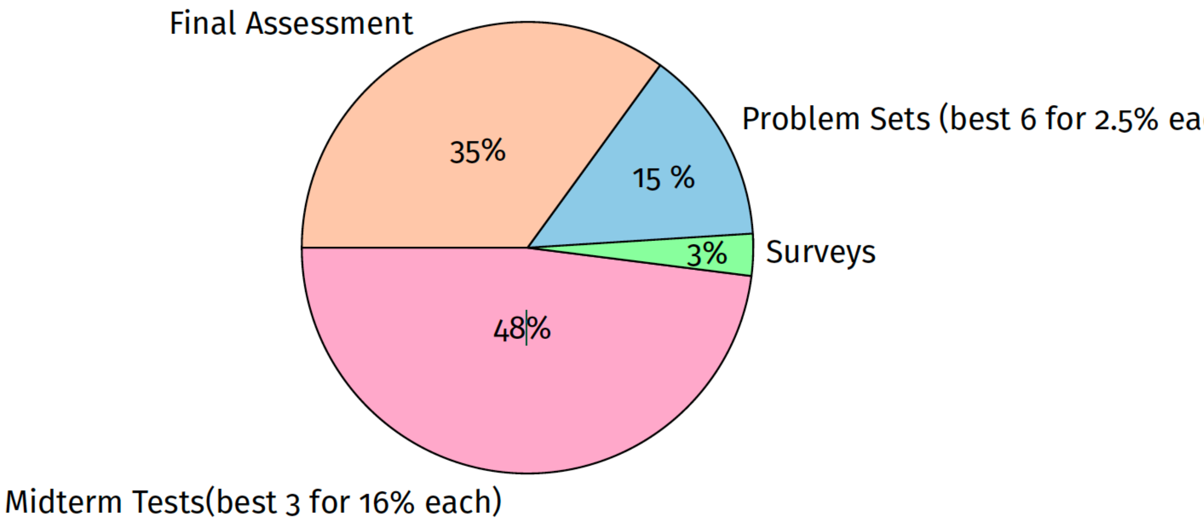
建议想学 237 的同学选 137！

<u>重要的时间节点：</u>

Sept 22.2021： DDL for 修改或添加课程
Feb 21.2022： DDL for drop class

考试时间（5 次）：
Term test1：Oct 21.2021
Term test2：Dec 3.2021
Term test3：Feb 3.2021
Term test4：Mar 17.2021
+ Final

课程构成



If one Term Test is held online, then 20% of its weight will be shifted evenly to problem sets.
If the final exam is held online then 30% of its weight will be shifted evenly to problem sets.

Note：8 个作业取 best 6/8，4 个考试取 best 3/4

注意：PS 的难度随课程时间推移逐渐增加，test 如果 online，PS 比重增加

非常宽松的评分要求意味着：

1. 要认真对待每一次 PS 和考试
2. 不要掉队，紧跟学习节奏
3. 不要作弊！不要作弊！不要作弊！

时间分配：

课前看视频（1h+）+ Lectures（3h）+ PS（3 小时+）+ Tutorial（1h）+ Studying（2h）

注意：PS 主要为证明题，但考试也有 computation problem，这些题需要自己去练

Tips：教授会发 practice problem，练习计算题；多看看 Piazza，教授和 TA 会解答

Set

Definition:

A set is any collection of **well-defined** and **distinct** objects.

The order of objects does not matter. eg. $\{2, 3, 8\}$ and $\{3, 8, 2\}$ are equivalent

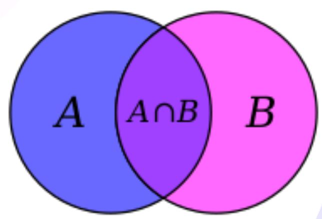
Notation:

$2 \in \{2,3,8\}$ $5 \notin \{2,3,8\}$ $\{2,3\} \subseteq \{2,3,8\}$ $\{1,2,3\} \not\subseteq \{2,3,8\}$

set builder:

$A = \{x \in \mathbb{R} \mid x \text{ is even}\}$ = The set of all real numbers x such that x is even

- The **naturals**[2] $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$,
- The **integers** $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$,
- The **rationals** $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z}, q \neq 0\}$,
- The **reals** $\mathbb{R}$ (the set of all infinite decimal expansion).



Intervals:

closed interval - [a, b]
open interval - (a, b)
half-open interval - [a, b), (a, b]
unbounded: $(-\infty, a), (a, \infty)$

Writing in set notation:

What are each of the following sets?

(a) $\mathbb{N} \cap \mathbb{Z} \cap \mathbb{Q} \cap \mathbb{R}$

(b) $\mathbb{N} \cup \mathbb{Z} \cup \mathbb{Q}$

(c) $(-1, e) \cup [0, \pi]$

(d) $(-1, e) \cap [0, \pi]$

(e) $(-e, \pi] \cap \mathbb{Z}$

(f) $(-1, 2] \cap \varnothing$

Logic

Definition:

P(x) is a proposition if it has a truth value associated to it

Quantifiers:

1) Universal quantifier $\forall$
2) Existential quantifier $\exists$

Conditional statements:

P -> Q means

注意：Quantifier 前后顺序很重要！有些情况不能改变！

Negation：

Statement	Negation
"A or B"	"not A and not B"
"A and B"	"not A or not B"
"if A, then B"	"A and not B"
"For all x, A(x)"	"There exist x such that not A(x)"
"There exists x such that A(x)"	"For every x, not A(x)"

**** 当括号前有 negation，拆开括号后$\forall$变成$\exists$，$\cap$变成$\cup$，P(x)变成$\neg \boldsymbol{P}(\boldsymbol{x})$**

What is $\neg(\forall x \in \mathbb{R}, \exists y \in A \text{ s.t. } x > y^2)$?

1. Negate each of the following statements without using any negative words ('no', 'not', 'none', etc):

   (a) "*Every page in this book contains at least one word whose first and last letters both come alphabetically before M.*"

   (b) "*I have a friend all of whose former boyfriends had at least two siblings with exactly three different vowels in their name.*"

   (c) "*If a student in this class likes the musical Cats then they are not my friend.*"

Implication and Truth Table:

P -> Q

Q -> P

P <-> Q

Converse vs Inverse vs Contrapositive

证明思路：

a. **直接证明(运用 definition)**

思路：Formalize 成数学语言，搞清楚 assumption 和 conclusion，写明 WTS，

$\forall x \in \mathbb{R}$，写 Let $x \in \mathbb{R}$

$\exists x \in \mathbb{R}$，写 Take $x = ?$

$P \rightarrow Q$，写 Assume P, WTS 写 Q

如果推不下去，看看有什么条件没用吗?

9. Write a formal proof for the following statement:

$$\forall a > 0,\ \exists b \in \mathbb{R} \text{ such that } (b + \sin b)a > 7.$$

b. **Prove by contraposition**

原命题如果不好证明，则使用 contrapositive，多用于 conditional statement 的证明

Example 1.2

Let x be a positive integer. If $x \neq 1$, then $x \neq x^2$.

Prove by contrapositive: Let $x \in \mathbb{Z}$. If $x^2 - 6x + 5$ is even, then x is odd.

c. **Prove by contradiction**

一般 negate 结论，然后推导出与假设矛盾的地方

Example: Prove that if x is a multiple of 6, then x is a multiple of 2

d. **Prove by induction**

Definition:

We define the set of natural numbers $\mathbb{N}$ = ______________________________

When we try to prove something that holds for all natural numbers, we can use mathematical induction, which consists of **base case** and **inductive step**.

Simple induction:

Strong induction:

Example: Prove by induction that for every positive integer n, the number $5^{2n} + 11$ is a multiple of 12.

Summary :

Problem set

可能是 137 中最劝退的部分，难度逐年增加（individual or group 1-2 人）

一般 2-4 题，需要花费大量时间

2021 Winter PS10

4. Let I be an open interval. Let f be a function defined on I. Let $a \in I$. Assume f is C^2 on I. Assume $f'(a) = 0$. As you know, a is a candidate for a local extremum for f. The "2nd Derivative Test" says that, under these circumstances:
   - IF $f''(a) > 0$, THEN f has a local minimum at a.
   - IF $f''(a) < 0$, THEN f has a local maximum at a.

   This theorem is easier to justify, and to generalize, using Taylor polynomials.

(a) Let P_2 be the 2nd Taylor polynomial for f at a. Write an explicit formula for P_2.

Using the idea that $f(x) \approx P_2(x)$ when x is close to a, write an intuitive explanation for the 2nd Derivative Test.

Note: We are not asking for a proof yet. Rather, we are asking for a short, simple, handwavy argument that would convince an average student that this result "makes sense" and "seems true".

(b) Now write an actual, rigorous proof for the 2nd Derivative Test. You will need to use the first definition of Taylor polynomial (the one in terms of the limit), the definition of limit, and the definition of local extremum.

Note: There are many other ways to prove the 2nd Derivative Test, but we want you to do it specifically this way. It will help with the next questions.

(c) What happens if $f''(a) = 0$? In that case, we look at $f^{(3)}(a)$; if it is also 0, then we look at $f^{(4)}(a)$, and we keep looking till we find one derivative that is not 0 at a.

More specifically, assume that f is C^n at a for some natural number $n \geq 2$ and that $f^{(n)}(a)$ is the smallest derivative that is not 0 at a. In other words, $f^{(k)}(a) = 0$ for $1 \leq k < n$ but $f^{(n)}(a) \neq 0$.

Using the same ideas as in Question 4a, complete the following statements, and give an intuitive explanation for them:

- IF ..., THEN f has a local minimum at a.
- IF ..., THEN f has a local maximum at a.
- IF ..., THEN f does not have a local extremum at a.

When you complete this, you will have come up with a new theorem. Let's call it the "Beyond-the-2nd-derivative Test". Make sure your theorem takes care of all possible cases.

Tests

Solutions, marking, and regrades

- Part A is a Quercus quiz.
  - Once it is over, you can see the answers you submitted and the answer key on Quercus.
  - The quiz is marked automatically. There are no regrades. If you do not understand one of the questions, use Piazza or office hours.
    For multiple-choice questions, every correct answer you select adds points, but every incorrect answer you select subtracts points (although the minimum score is 0 on each question). It is possible to select some correct answers and still get 0 on a question.
- Part B (long answer) is on Quercus.
  - After the test, we will post sample solutions and lists of common errors below.
  - If you want to request a regrade, follow same instructions as for assignments.
- Coverage: Test X will include Units xx-xx. To prepare:
  - Re-watch the videos
  - Solve all the practice problems
  - Review class questions and homework assignments
- Structure: Test X will have two parts:
  - Part A (90 minutes): a quiz on Quercus. Some questions will be multiple choice; other questions will require a numerical answer.
  - Part B (120 minutes): long-answer questions (mostly proofs) on Crowdmark.
  - If this were a regular in-person test, we would give you only 60 minutes for each part. That is how long we think it should actually take you. We are giving you additional time to accommodate for any possible technical problems you may have, and to give you time to scan and upload to Crowdmark. If you choose to leave uploading for the last minute and don't complete it on time, that was your choice.

相比 PS，反而比较简单

- Part A，多为计算题（填空或选择，考察计算能力，必须写过程）
- Part B，2-3 题证明题（多考察 definition 和数学思维）

2. *[6 points total; 2 points per part]*
   Calculate the following limits (or explain that they do not exist):

(a) $\lim_{x\to 3} \frac{\sin(3x)}{x}$

Your answer: $(\sin 9)/3$

Solution:

$$\lim_{x\to 3} \frac{\sin(3x)}{x} = \frac{\lim_{x\to 3} \sin(3x)}{\lim_{x\to 3} x} = \frac{\sin 9}{3}.$$

(b) $\lim_{x\to -\infty} \frac{x^2+2}{3x^2+4}$

Your answer: $1/3$

Solution:

$$\lim_{x\to -\infty} \frac{x^2+2}{3x^2+4} = \lim_{x\to -\infty} \frac{1+2/x^2}{3+4/x^2} = \frac{\lim_{x\to -\infty}(1+2/x^2)}{\lim_{x\to -\infty}(3+4/x^2)} = \frac{1+0}{3+0} = \frac{1}{3}.$$

(c) $\lim_{x\to 0} \frac{\sin^2(3x^3)}{x^6}$

Your answer: 9

Solution:

$$\lim_{x\to 0} \frac{\sin^2(3x^3)}{x^6} = \lim_{x\to 0} \left(\frac{\sin(3x^3)}{3x^3}\right)^2 \frac{(3x^3)^2}{x^6} = \lim_{x\to 0} \left(\frac{\sin(3x^3)}{3x^3}\right)^2 \cdot 3^2 = 1^2 \cdot 3^2 = 9$$

Proof 和背 definition 的能力：

8. *[5 points]* Write a proof for the following theorem:

Theorem Let $a \in \mathbb{R}$. Let f be a function defined on all of $\mathbb{R}$, except maybe a. Assume that $f(x)$ is never 0.

- IF $\lim_{x \to a} f(x) = \infty$
- THEN $\lim_{x \to a} \frac{1}{f(x)} = 0$

Write a formal proof directly from the definitions of limit. Do not use the limit laws.

Solution. WTS: $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x$ satisfying $0 < |x - a| < \delta \implies |1/f(x) - 0| < \epsilon$.

- Let us fix $\varepsilon > 0$.
- Use $M = \frac{1}{\varepsilon}$ in the definition of $\lim_{x \to a} f(x) = \infty$. There must exist $\delta > 0$ such that

$$\forall x \in \mathbb{R}, \quad 0 < |x - a| < \delta \implies f(x) > \frac{1}{\varepsilon}.$$

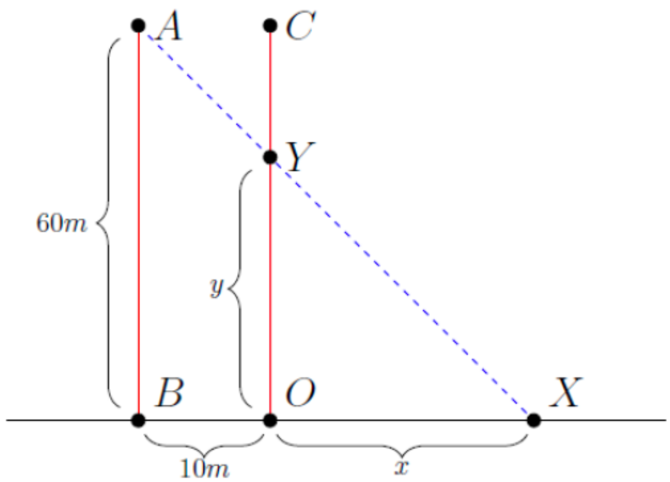
- This is the value of δ I need.
- Let's verify it works. Take $x \in \mathbb{R}$ and assume $0 < |x - a| < \delta$.
  Then $f(x) > \frac{1}{\varepsilon} > 0$,
  therefore $0 < \frac{1}{f(x)} < \varepsilon$,
  and $\left| \frac{1}{f(x)} \right| < \varepsilon$, which is what we needed.

英文阅读+建模+画图能力：

4. *[5 points]* Two poles, 60 meters high, stand next to each other, 10 meters apart. At the top of one pole is a light. At the top of the other pole there is an engineering student who drops down a watermelon. The light on the first pole illuminates the watermelon and casts a shadow on the floor. How fast is the shadow of the watermelon moving when the watermelon is halfway to the ground and falling at a speed of 15 m/s?

Solution.

1) Modelling:



The two red lines in the figure are the poles: AB and CO. The light is at A. The watermelon is falling from C to O and at a given time it is at Y. The light from A to the watermelon (Y) produces a shadow at the point X on the floor. In the picture, the distances x and y depend on time. The other distances do not change over time. The position of the shadow is given by x.

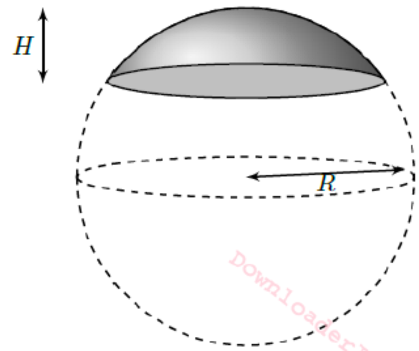
The two right triangles ABX and YOX are similar, and so we have

$$\frac{60}{10+x} = \frac{y}{x}. \tag{1}$$

Moreover, at the given time we have $y = 60/2 = 30\,\mathrm{m}$, $\dfrac{dy}{dt} = -15\,\mathrm{m/s}$ (where the minus sign is due to the fact that y is decreasing in time). At this given time, we want to compute $\dfrac{dx}{dt}$.

空间想象力:

4. *[4 points]* I have a spherical orange with radius R. I cut a piece from the top that has height H, like in the picture:



Calculate the volume of the piece. Justify your answer.

Your answer: $\frac{1}{3}\pi H^2(3R - H)$

Note: When $H = 2R$, this gives $\frac{4}{3}\pi R^3$ which is the volume of a sphere.
When $H = R$ this gives $\frac{2}{3}\pi R^3$, which is the volume of a hemisphere.

Solution:

We can think of this piece as a solid of revolution:

- Consider the circle centered at the origin with radius R. The equation is $x^2 + y^2 = R^2$.
- Consider the region in red in the picture below. It is the region between the y-axis and the right-side of the circle ($x = \sqrt{R^2 - y^2}$), from $y = R - h$ to $y = R$).
- We want to compute the volume of the solid obtained by rotating the red region around the y-axis.