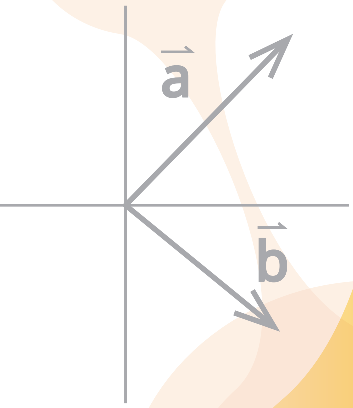
MAT223H

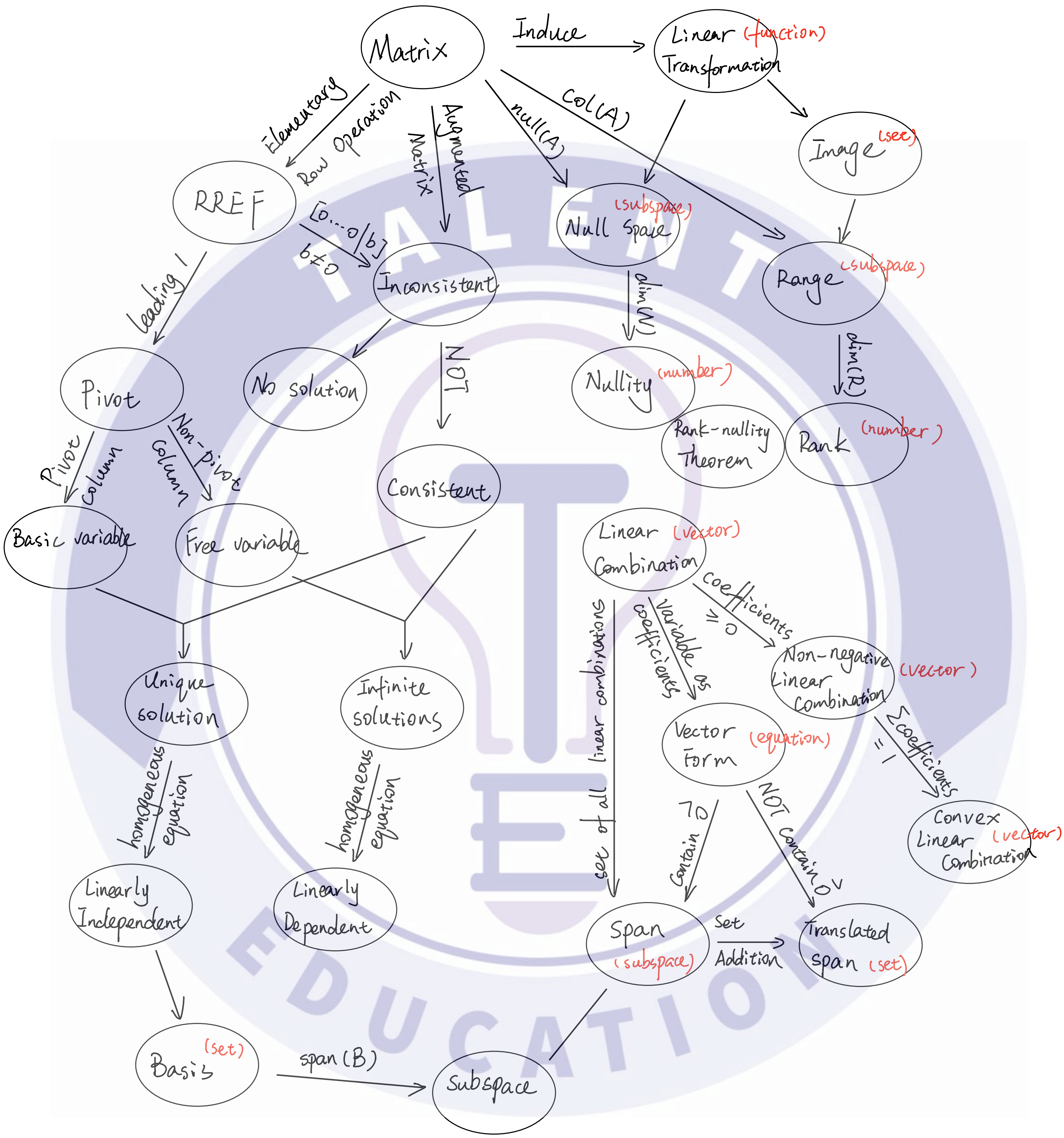
Linear Algebra I

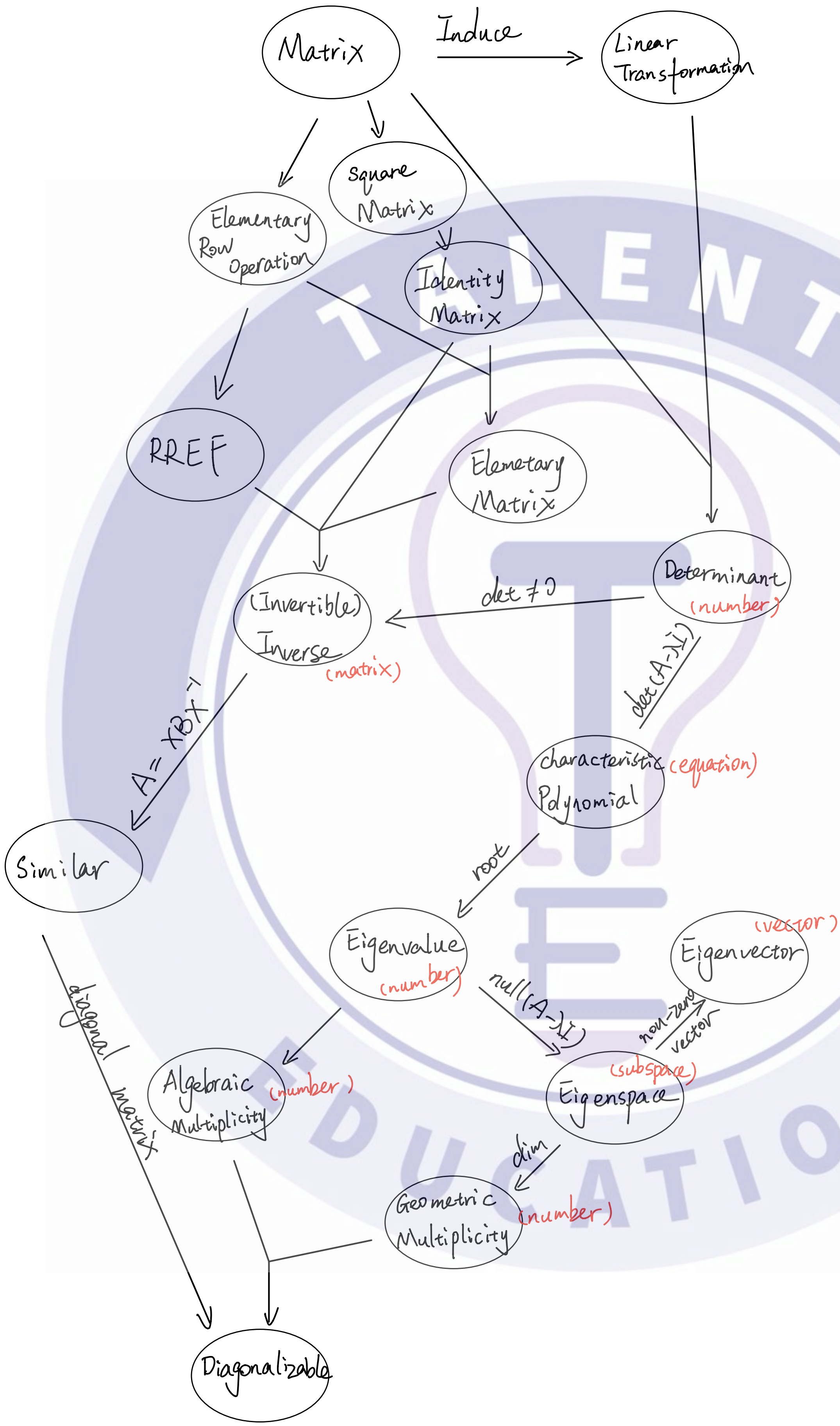


Fall 2021

Week 1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$


Matrix
Induce
Linear (function) Transformation
Image (set)
Elementary Row Operation
Augmented Matrix
null(A)
Col(A)
RREF
[0...0|b] b≠0
Inconsistent
Null Space (subspace)
Range (subspace)
leading 1
No solution
NOT
dim(N)
dim(R)
Pivot
Nullity (number)
Rank-nullity Theorem
Rank (number)
Pivot Column
Non-pivot Column
Consistent
Basic variable
Free variable
Linear (vector) Combination
coefficients ≥ 0
variable as coefficients
set of all linear combinations
Non-negative Linear Combination (vector)
Unique solution
Infinite solutions
Vector Form (equation)
Σcoefficients = 1
homogeneous equation
homogeneous equation
Contain 0
NOT Contain 0
Convex Linear Combination (vector)
Linearly Independent
Linearly Dependent
Span (subspace)
Set Addition
Translated span (set)
Basis (set)
span(B)
Subspace


Matrix
Induce
Linear Transformation
Square Matrix
Elementary Row Operation
Identity Matrix
RREF
Elemetary Matrix
Determinant
(number)
det ≠ 0
(Invertible) Inverse
(matrix)
det(A-λI)
characteristic Polynomial
(equation)
A = XBX⁻¹
Similar
root
Eigenvalue
(number)
Eigenvector
(vector)
null(A-λI)
non-zero vector
Eigenspace
(subspace)
diagonal matrix
Algebraic Multiplicity
(number)
dim
Geometric Multiplicity
(number)
Diagonalizable

SYSTEMS OF LINEAR EQUATIONS

Systems of Linear Equations and Elimination

A **linear equation** in the variables $x_1, \cdots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where b and the **coefficients** $a_1, \cdots, a_n$ are real or complex numbers, usually known in advance.

A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables.

A **solution** of the system is a list $(s_1, \cdots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \cdots, s_n$ are substituted for $x_1, \cdots, x_n$, respectively. The set of all possible solutions is called the **solution set** of the linear system.

PRACTICE QUESTIONS

1. Find the solution of the linear equation

$$2x_1 + 3x_2 = 9$$

$$x_1 - 2x_2 = 1$$

The Matrix Representation of a System of Linear Equations

The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**. Given the system

$$x_1 - 2x_2 + x_3 = 0$$

$$2x_2 - 8x_3 = 8$$

$$5x_1 - 5x_3 = 10$$

with the coefficients of each variable aligned in columns, the matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$

is called the **coefficient matrix** (or **matrix of coefficients**) of the system, and

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

is called the **augmented matrix** of the system.

Elementary Row Operation

1. (Scaling) Multiply all entries in a row by a nonzero constant.
2. (Interchange) Interchange two rows.
3. (Replacement) Replace one row by the sum of itself and a multiple of another row.

Row Echelon Form

A nonzero row or column in a matrix means a row or column that contains at least one nonzero entry. A **leading entry** of a row refers to the leftmost nonzero entry (in a nonzero row).

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

PRACTICE QUESTION

2. Indicate the following matrices are in row echelon form or reduced row echelon form or nothing.

(i) $\begin{bmatrix} 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(vii) $\begin{bmatrix} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$ (viii) $\begin{bmatrix} 1 & 0 & 3 & 5 & 10 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$ (ix) $\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Pivot Positions

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

A **pivot** is a nonzero number in a pivot position that is used as needed to create zeros via row operations.

PRACTICE QUESTIONS

3. Find the solutions to the following system of linear equations using row reductions.

$$2x - y + z = 3$$

$$2x + 3y - 6z = -4$$

$$-x - 2y + 5z = 4$$

Consistent Systems and Unique Solutions

A system of linear equation is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.

The variables corresponding to pivot columns in the matrix are called **basic variables**. The other variables, are called **free variables** where free means that you are free to choose any value for the variable.

Theorem

A linear system is consistence if and only if the rightmost column of the augmented matrix is not a pivot column. That is, if and only if an echelon form of the augmented matrix has no row of the form

$$[0 \ \cdots 0 \ b] \qquad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

PRACTICE QUESTIONS

4. How many free variable columns does each augmented matrix have?

a. $\left[\begin{array}{cc|c} 1 & 8 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

b. $\left[\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & -7 \\ 0 & 0 & 0 \end{array}\right]$

c. $\left[\begin{array}{ccc|c} 1 & 3 & -6 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right]$

5. The reduced row-echelon forms of the augmented matrices of four systems are given below. How many solutions does each system have?

a. $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 14 \\ 0 & 0 & 1 & -12 \end{array}\right]$

c. $\left[\begin{array}{cc|c} 1 & 0 & 19 \\ 0 & 1 & -6 \end{array}\right]$

d. $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(Winter 2019 TT1 Q2) Consider the system $\begin{cases} x_1 + 2x_2 - 3x_3 = b_1 \\ 2x_1 + 4x_2 - 6x_3 + x_4 = b_2 \\ 6x_1 + 13x_2 - 17x_3 + 4x_4 = b_3 \end{cases}$.

In this problem you may use the fact that $\operatorname{rref}\left(\begin{bmatrix} 1 & 2 & -3 & 0 & -5 \\ 2 & 4 & -6 & 1 & -8 \\ 6 & 13 & -17 & 4 & -21 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -5 & 0 & -7 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

(a) (2 points) Write the complete solution to the system when $b_1 = -5$, $b_2 = -8$, $b_3 = -21$. Express your answer in vector form.

(b) (2 points) Are there values of b_1, b_2, b_3 that make the system inconsistent? If so, given an example of such values. If not, explain why not.